Lecture 5

Friday, October 11, 2019 6:21 AM

• Discuss Cantor's Thm from Lecture 4 notes.

Compactness. (As usual, (X, d) is a metric space.)

Def. A subset $K \subseteq X$ is compact if any open cover $\{G_\alpha\}_{\alpha \in I}$ of K — $K \subseteq \bigcup_{\alpha \in I} G_\alpha$ and $G_\alpha \subseteq X$ open — has a finite subcover: $G_{\alpha_1}, \dots, G_{\alpha_n}$ w/ $K \subseteq \bigcup_{k=1}^{n} G_{\alpha_k}$

Basic Props: (i) If $K \subseteq X$ is compact $\Rightarrow$ K is closed.
(ii) If $K \subseteq X$ is compact, $F \subseteq K$ is closed $\Rightarrow$ F is compact.

These require proofs, left as Ex / read Conway.

Thm1. If X is compact $\Rightarrow$ X is complete.

For Pf we need:

Prop1. $K \subseteq X$ is compact $\Leftrightarrow$ $\forall \{F_\alpha\}_{\alpha \in I}$ of closed sets $F_\alpha \subseteq K$ w/ FIP — for any finite subcollection $F_{\alpha_1}, \dots, F_{\alpha_n}$, $\bigcap_{k=1}^{n} F_{\alpha_k} \neq \emptyset$ — it holds that $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.

Pf. $\Rightarrow$: Let $\{F_\alpha\}$ be collection of closed subsets $F_\alpha \subseteq K$ w/ FIP. Suppose $\bigcap_{\alpha \in I} F_\alpha = \emptyset$. Consider open sets $G_\alpha = X \setminus F_\alpha \Rightarrow \bigcup_{\alpha \in I} G_\alpha = X \setminus \bigcap$

In particular $\{G_\alpha\}$ is open cover of K. But K compact $\Rightarrow \exists$ finite subcover $G_{\alpha_1}, \dots, G_{\alpha_n}$, i.e. $K \subseteq \bigcup_{k=1}^{n} G_{\alpha_k} = X \setminus \bigcap_{k=1}^{n} F_{\alpha_k}$. Since $F_{\alpha_j} \subseteq K$ it must then be that $\bigcap_{k=1}^{n} F_{\alpha_k} = \emptyset$, which $\times$ FIP. Thus $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.

$\Leftarrow$: Let $\{G_\alpha$ be open cover of K. Consider $\underbrace{F_\alpha = K \cap (X \setminus G_\alpha)}_{\text{closed subsets}} \subseteq K$.

Suppose there is no finite subcover. Then, for any $\alpha_1, \dots, \alpha_n$, $\exists x_0 \in K \setminus \bigcup_{k=1}^{n}$ i.e. $x_0 \in K$ and $x_0 \in X \setminus \bigcup_{k=1}^{n} G_{\alpha_k}$ $\Rightarrow$ $x_0 \in \bigcap_{k=1}^{n} F_{\alpha_k}$, i.e. $\{F_\alpha\}$ has FIP.

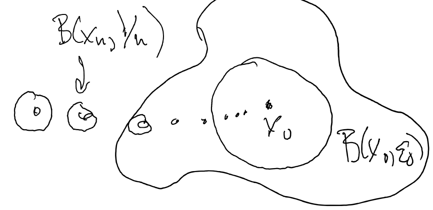
assumption $\bigcap_\alpha F_\alpha \neq \emptyset$. $\Rightarrow$ $K \not\subseteq$ G_α which $\times$. Thus, $G_\alpha\}$ has finite subcover $\Rightarrow$ K is compact.

Def. A metric space X is sequentially compact if every sequence has a convergent subsequence.

Lebesgue's Covering Lemma. Assume X is sequentially compact. If $G_\alpha\}_{\alpha \in I}$ is an open cover $\Rightarrow \exists \varepsilon > 0$ s.t. $\forall x \in X$, $B(x, \varepsilon) \subseteq G_\alpha$ for some α.

Pf. Suppose no such $\varepsilon > 0$ exists. Then, $\forall n \in \mathbb{N}$ $\exists x_n \in X$ s.t. $B(x_n, \frac{1}{n}) \not\subseteq G_\alpha$ ... assumption. $\exists$ subsequence

Pf. Suppose no such $\varepsilon > 0$ exists. Then, $\forall n \in \mathbb{N}$ $\exists x_n \in X$ s.t. $B(x_n, 1/n) \not\subseteq G_\alpha$ for any α. assumption, $\exists$ subsequence $\}_{k=1}^{\infty}$ that converges to, say, $x_0 \in X$. Well, $x_0 \in G_{\alpha_0}$ some α_0, and $\exists\, \varepsilon_0 > 0$ s.t. $B(x_0, \varepsilon_0) \subseteq G_{\alpha_0}$.



$x_{n_k} \to x_0 \Rightarrow$

$\exists L$ s.t. $d(x_{n_k}, x_0) < \varepsilon_0/2$ and $\frac{1}{n_k} < \varepsilon_0/2$ for $k \geq L$.

Thus, for any $y \in B(x_{n_k}, 1/n_k)$ we have for $k \geq L$

$$d(y, x_0) \overset{\Delta\text{-ineq}}{\leq} d(y, x_{n_k}) + d(x_{n_k}, x_0) < \frac{1}{n_k} + \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0$$

But then $y \in B(x_0, \varepsilon_0) \subseteq G_{\alpha_0} \overset{y \text{ arbitr.}}{\Rightarrow} B(x_{n_k}, 1/n_k) \subseteq G_{\alpha_0}$ $\times$ assumption above. $\boxtimes$